Complex Mechanics：

A Unified Approach to Classical and Quantum Mechanics

複數力學：邁向古典力學與量子力學的統一

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Abstract

Aims: To familiarize the researchers with complex mechanics and to pave the way for them to analyze quantum systems by using methods already developed in classical mechanics.

Method: Quantum mechanics is an experimental science defined in the real world, but understanding the how and the why of quantum mechanics requires a viewpoint from the complex space. It may be said that quantum mechanics is intrinsically a complex-valued science. The complex space is where a quantum motion takes place, while the real space is where we take the measurement of the motion. Quantum mechanics lays out the distribution and the evolution of the measurement data, while the complex-valued mechanics introduced here describes the quantum motion in the complex space before it is measured. In the complex space, quantum motions are deterministic so that all the methods of classical mechanics can be applied. The projection of the complex-valued motion into the real space recovers and confirms the quantum phenomena observed from the measurement data.

Results: The complex mechanics, a unified approach to classical and quantum mechanics, provides a bridge between the probabilistic interpretation in the real space and the deterministic interpretation in the complex space. Through this bridge, researchers in classical mechanics can employ methods familiar to them to analyze quantum systems in the complex space, and to predict and verify various quantum phenomena by projecting the results of analysis into the real space.

Keywords: Quantum Mechanics, Classical mechanics, Complex Mechanics

Introduction

The lack of dynamic equations of motion with respect to time, which is one of the sources of controversies of quantum mechanics, makes it impossible to analyze stability, chaos, bifurcation, and many other nonlinear features existing in quantum systems. This impossibility has long been taken for granted due to a common belief that the probabilistic nature of quantum phenomena is in no way described or represented by deterministic nonlinear models. However, probabilistic viewpoint and deterministic viewpoint may not be as conflicting as we commonly think. Consider a scenario that a dynamic motion occurs in the complex space but only the real-part of the motion can be measured. Due to the influence of the unmeasurable imaginary-part motion and its interaction with the real-part motion, the measured real-part motion is uncertain and can only be described probabilistically. On the other hand, the same motion, if viewed from the complex space, is governed uniquely by a set of complex-valued nonlinear equations, which are totally deterministic. In such a situation, probabilistic interpretation and deterministic interpretation can be equally applied to the same motion, depending on which space we view from.

In quantum mechanics, we encounter the same scenario that actual particle motions occur in the complex space, but what we sense and measure are merely the real parts of the motions as depicted in Fig. 1, which give rise to what we call quantum phenomena. The correctness of this scenario describing quantum world is easy to verify by solving the complex-valued nonlinear equations of motion, projecting the solutions into the real space and then comparing with the measurement data. In recent years, an excellent consistency of the projected solutions with the various quantum effects has been justified and reported under the framework of complex-valued mechanics [1,2]. The present paper aims to familiarize the researchers with complex mechanics to pave the way for them to analyze quantum systems by using methods already developed in classical mechanics.

Complex-valued Quantum Dynamics:

By extending canonical variables $(\mathbf{q}, \mathbf{p})$ to a complex domain, it can be shown [1] that the complex-valued $\mathrm{Schrödinger}$ equation

$$\mathrm{i}\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi \,, \tag{1}$$

turns out to be the quantum Hamilton-Jacobi equation

$$\partial S / \partial t + H(t, \mathbf{q}, \mathbf{p})\big|_{\mathbf{q}=\nabla S} = 0 \tag{2}$$

by the transformation $\Psi = \exp(\mathrm{i}S / \hbar)$. The quantum Hamiltonian in Eq.(2) is found to be

$$H(t, \mathbf{q}, \mathbf{p}) = \frac{1}{2m} \mathbf{p}^2 + V(t, \mathbf{q}) + Q(\Psi(t, \mathbf{q})) \,, \tag{3}$$

where Q is the intrinsic quantum potential defined by

$$Q(\Psi(t,\mathbf{q})) = \frac{\hbar}{2m\mathrm{i}} \nabla \cdot \mathbf{p}\Big|_{\mathbf{p}=\nabla S} = \frac{\hbar}{2m\mathrm{i}} \nabla^2 S = -\frac{\hbar^2}{2m} \nabla^2 \ln \Psi(t,\mathbf{q})\,. \tag{4}$$

With the Hamiltonian given by Eq.(3), quantum Hamilton equations of motion in the quantum state $\Psi(t,\mathbf{q})$ become

$$\frac{d\mathbf{q}}{dt} = -\frac{\partial H(t,\mathbf{q},\mathbf{p})}{\partial \mathbf{p}} = \frac{1}{m}\mathbf{p}\,, \tag{5a}$$

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H(t,\mathbf{q},\mathbf{p})}{\partial \mathbf{q}} = -\frac{\partial}{\partial \mathbf{q}}\left[V(t,\mathbf{q}) - \frac{\hbar^2}{2m}\nabla^2 \ln \Psi(t,\mathbf{q})\right], \tag{5b}$$

where the canonical momentum $\mathbf{p}$ can be determined from the action function S by the canonical transformation as

$$\mathbf{p} = \nabla S(t,\mathbf{q}) = -\mathrm{i}\hbar \nabla \ln \Psi(t,\mathbf{q}) \in \mathbb{C}^n\,. \tag{6}$$

This expression for $\mathbf{p}$ is actually a complex extension of the Bohm's guidance condition [3]. The complex momentum function $\mathbf{p}$ is a realization of the momentum operator $\hat{\mathbf{p}}$ in the complex space according to the definition $\hat{\mathbf{p}}\Psi = \mathbf{p}\Psi$, which together with Eq.(4) gives an expression for the momentum operator as $\hat{\mathbf{p}} = -\mathrm{i}\hbar\nabla$. Similarly, the quantum operator $\widehat{Q}$ for an arbitrary quantum observable Q can be identified from its complex-space realization $Q(t,\mathbf{q},\mathbf{p})$ via the relation $\widehat{\mathbf{Q}}\Psi = \mathbf{Q}\Psi$ [4]. For instance, the x component L_x of the complex angular momentum $\mathbf{L} = \mathbf{q} \times \mathbf{p}$ for $\mathbf{q} = [x\ y\ x] \in \mathbb{C}^3$ and $\mathbf{p} = [p_x\ p_y\ p_z] \in \mathbb{C}^3$ can be computed as

$$L_x = yp_z - zp_y = y\left(\frac{\hbar}{\mathrm{i}}\frac{\partial \ln \psi}{\partial z}\right) - z\left(\frac{\hbar}{\mathrm{i}}\frac{\partial \ln \psi}{\partial y}\right) = \frac{-\mathrm{i}\hbar}{\psi}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\psi\,. \tag{7}$$

Comparing the above equation to the definition $L_x\psi = \hat{L}_x\psi$ gives the operator $\hat{L}_x$ as

$$\widehat{L}_x = -\mathrm{i}\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) = y\widehat{p}_z - z\widehat{p}_y\,. \tag{8}$$

The combination of Eq.(5a) with Eq.(6) gives the complex equation of motion in the state $\Psi(t,\mathbf{q})$ as

$$\dot{\mathbf{q}} = -\mathrm{i}(\hbar / m)\nabla \ln \Psi(t,\mathbf{q}), \quad \mathbf{q} \in \mathbb{C}^n\,. \tag{9}$$

For a 1D stationary state with coordinate $q = x \in \mathbb{C}$, a solution of the Schrödinger's equation can be expressed by $\Psi(x,t) = \psi(x)e^{-\mathrm{i}(E/\hbar)t}$ from which Eq.(9) reduces to

$$\frac{dx}{dt} = f(x) = -\frac{\mathrm{i}\hbar}{m}\frac{1}{\psi(x)}\frac{d\psi}{dx}, \quad x \in \mathbb{C}\,. \tag{10}$$

The related complex action function S becomes $S(x,t) = -\mathrm{i}\hbar \ln \psi(x) - Et$, and the quantum Hamilton-Jacobi equation (2) then reads

$$H(x,p,t) = \frac{1}{2m}p^2 + V(x) - \frac{\hbar^2}{2m}\frac{d^2 \ln \psi}{dx^2} = -\frac{\partial S}{\partial t} = E\,. \tag{11}$$

This is an explicit energy conservation law in quantum mechanics, which manifests that the total energy in a quantum system comprises three terms: the kinetic energy $p^2/(2m)$, the external potential energy $V(x)$, and the intrinsic potential energy $Q(x) = -\hbar^2/(2m)d^2\psi/dx^2$. With p

given by Eq.(6), the energy conservation law (11) turns out to be the time-independent $\mathrm{Schrödinger}$ equation

$$\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+\left(E-V(x)\right)\psi(x)=0\,. \tag{12}$$

The quantum potential Q has an intimate relation to the probability interpretation of the standard quantum mechanics [5]. From the energy conservation law (11), we can express the total potential as

$$V_{\mathrm{Total}}=Q+V=E-\frac{1}{2m}p^2=E+\frac{\hbar^2}{2m}\left(\frac{d\ln\psi}{dx}\right)^2. \tag{13}$$

From Eqs.(5), a succinct expression for the quantum Newton second law becomes

$$m\ddot{x}=-\frac{dV_{\mathrm{Total}}}{dx}=-\frac{\hbar^2}{2m}\frac{d}{dx}\left(\frac{d\ln\psi}{dx}\right)^2. \tag{14}$$

Since this equation of motion is independent of the constant E, we can choose $E=0$ as the reference energy level for V_{Total}. The magnitude of the total potential now has the expression

$$\left|V_{\mathrm{Total}}\right|=\left|\frac{\hbar^2}{2m}\left(\frac{d\ln\psi}{dx}\right)^2\right|=\frac{\hbar^2}{2m}\frac{\left|d\psi/dx\right|^2}{\psi^*\psi}, \tag{15}$$

which states that the magnitude of the total potential is inversely proportional to $\psi^*\psi$. A spatial point with large value of $\psi^*\psi$ corresponds to a location with law potential barrier and hence a large accessibility to this point. This fact legitimates the use of $\psi^*\psi$ as the probability measure for a particle to appear at a specified spatial point.

Chaos and Multiple Paths

Chaotic behavior of a nonlinear system is featured by the high sensitivity of its phase-space trajectories to its initial conditions. Because the trajectory of a particle is not a well-defined quantity in quantum mechanics, the manifestations of chaotic motion in quantum mechanics remain a controversial issue. This difficulty can be conquered by employing the nonlinear quantum dynamics introduced above. For example, we consider a 1D quantum system with coordinate $x=x_R+\mathrm{i}x_I\in\mathbb{C}$, whose equations of motion for the real and imaginary parts are given by Eq.(10) as

$$\frac{dx_R}{dt}=\mathrm{Re}\left(\frac{\hbar}{\mathrm{i}m\psi}\frac{\partial\psi(t,x)}{\partial x}\right)=f_R(x_R,x_I,t),\qquad \frac{dx_I}{dt}=\mathrm{Im}\left(\frac{\hbar}{\mathrm{i}m\psi}\frac{\partial\psi(t,x)}{\partial x}\right)=f_I(x_R,x_I,t)\,. \tag{16}$$

For a given initial position $x(0)=x_R(0)+\mathrm{i}x_I(0)$, a unique trajectory can be found from Eq.(16) on the complex plane. However, the uniqueness of trajectory on the complex plane does not imply the uniqueness of trajectory on the real axis, since different points on the complex plane may be projected into the same point on the real axis. By fixing the real-part initial position $x_R(0)=x_R^0$ and letting $x_I(0)$ vary in $\mathbb{R}$, a set of complex trajectories can be determined from Eq.(16):

$$\Omega = \left\{ (x_R(t), x_I(t)) \middle| \dot{x}_R = f_R(x_R, x_I, t),\ \dot{x}_I = f_I(x_R, x_I, t),\ x_R(0) = x_R^0,\ x_I(0) \in \mathbb{R} \right\}. \tag{17}$$

The projection of Ω onto the real axis gives rise to a set of real trajectories,

$$\Omega_R = \left\{ x_R(t) \middle| \dot{x}_R = f_R(x_R, x_I, t),\ \dot{x}_I = f_I(x_R, x_I, t),\ x_R(0) = x_R^0,\ x_I(0) \in \mathbb{R} \right\}, \tag{18}$$

as if they were all originated from the same initial position $x_R(0) = x_R^0$. Since only the real part $x_R(t)$ can be measured, what we have observed are the trajectories in Ω_R, which comprises an infinite number of real trajectories all emerging from the same initial position $x_R(0) = x_R^0$. This is just the multi-path phenomenon considered in Feynman's fractal space-time approach to quantum mechanics [6,7]. Classical chaos analysis is to consider the sensitivity of $x_R(t)$ to the initial perturbation $x_R^0 \to x_R^0 + \delta x_R^0$. One of the distinct features of quantum chaos is that even if the initial perturbation δx_R^0 is zero, the trajectory $x_R(t)$ still perturbs and diverges. We call this phenomenon "strong chaos" [8], because the chaos behavior is so severe that the trajectory diverges spontaneously without any perturbation in the initial position. All of such perturbed trajectories with $\delta x_R^0 = 0$ are caused by the unobservable $x_I(0)$ as shown schematically in Fig. 2. The analysis of strong chaos then amounts to considering the trajectory divergence in Ω_R due to the variation of $x_I(0)$.

Stability and Quantization

Stability of quantum systems can be analyzed by extending the Lyapunov stability theory to a complex domain. Let x_e be an equilibrium point (fixed point) satisfying $\dot{x} = f(x_e) = 0$. Then we say that the quantum system (10) is Lyapunov stable at x_e, if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$|x_0 - x_e| < \delta \quad \Rightarrow \quad |x(t, x_0) - x_e| < \varepsilon, \quad \forall t \ge t_0, \tag{19}$$

where $|x_0 - x_e|$ denotes the magnitude of the complex number $x_0 - x_e$. The system (10) is further said to be asymptotically stable at x_e if it is Lyapunov stable, and $x(t)$ approaches x_e as $t \to \infty$. It can be shown [9] that the asymptotic convergence is ensured, if the derivative of $f(x)$ at $x = x_e$ satisfies

$$\mathrm{Re}(\partial f / \partial x)_{x_e} < 0. \tag{20}$$

According to the value of $f'(x_e) = \alpha + \mathrm{i}\beta$, we can classify equilibrium points of a quantum system into several types: (1) a node with $\alpha \ne 0$, $\beta = 0$; (2) A center with $\alpha = 0$, $\beta \ne 0$; (3) a focus with $\alpha \ne 0$, $\beta \ne 0$. It is interesting to find that saddle points do not appear as equilibrium points in a quantum system, but appear as singular points [9].

In standard quantum mechanics, the quantization of $f(x,p)$ in a given quantum state $\psi(x)$ is manifested via the expectation

$$\langle f(x,p) \rangle_\psi \equiv \langle \psi | f(\hat{x}, \hat{p}) | \psi \rangle = \int_{-\infty}^{+\infty} \psi^*(x) f\left(x, -\mathrm{i}\hbar \frac{\partial}{\partial x}\right) \psi(x) dx. \tag{21}$$

Under the framework of complex-valued mechanics, $f(x,p)$ is a function of time with x and p

solved from Eq.(10). Hence, taking time average is preferable to the mean value of $f(x,p)$:

$$\langle f(x,p)\rangle_{x(t)} = \frac{1}{T}\int_0^T f(x(t),p(t))\ dt\,, \tag{22}$$

where T is the period of the trajectory $x(t)$. Via the relation (10), we can replace the time integration in Eq.(22) by a contour integration along the complex trajectory $x(t)$ denoted by c:

$$\langle f(x,p)\rangle_{x(t)} = \frac{m\mathrm{i}}{T\hbar}\oint_c f\left(x,-\mathrm{i}\hbar\frac{d\ln\psi(x)}{dx}\right)\frac{\psi(x)}{\psi'(x)}\ dx = \frac{m\mathrm{i}}{T\hbar}\oint_c f(x)\frac{\psi(x)}{\psi'(x)}\ dx\,. \tag{23}$$

Quantization is just the phenomenon that $\langle f(x,p)\rangle_{x(t)}$ appears to be independent of the trajectory $x(t)$ [9]. Let $\Omega(\psi,T)$ be the set consisting all the trajectories satisfying Eq.(10) with a period T:

$$\Omega(\psi,T) \triangleq \left\{x(t)\left|\frac{dx}{dt} = \frac{\hbar}{\mathrm{i}m}\frac{1}{\psi(x)}\frac{d\psi(x)}{dx},\ x(t+T)=x(t)\right.\right\}. \tag{24}$$

The time-averaged value $\langle f(x,p)\rangle_{x(t)}$ is said to be locally stationary (or locally trajectory-independent) in the set $\Omega(\psi,T_1)$, if the average of $f(x,p)$ taken along all the trajectories in $\Omega(\psi,T_1)$ is a fixed value

$$\langle f(x,p)\rangle_{x(t)} = f_1 = \text{constant},\quad \forall x(t)\in\Omega(\psi,T_1)\,. \tag{25}$$

In the same way, we can define another set $\Omega(\psi,T_2)$ within which $\langle f(x,p)\rangle_{x(t)}$ is locally stationary at the value f_2. Then, the collection of all the stationary values of $\langle f(x,p)\rangle_{x(t)}$ in an increasing order $\{f_1,\ f_2,\ f_3,\ \cdots\}$ establishes a quantization of $\langle f(x,p)\rangle_{q(t)}$ with f_i denoting its i^{th} level in the state ψ. If $\langle f(x,p)\rangle_{x(t)}$ has only one quantized level, $\langle f(x,p)\rangle_{x(t)}$ is said to be conservative in the state ψ, i.e., $\langle f(x,p)\rangle_{x(t)}$ is globally trajectory-independent in ψ.

Bifurcation and Quantum Entanglement

The term bifurcation is used to describe any sudden change in the dynamics of nonlinear systems, and the bifurcation theory is mainly concerned with the study of how the character of equilibrium points changes as parameters of a nonlinear system change. However, this concept of bifurcation cannot be conveyed directly to quantum mechanics, where it is impossible to talk about fixed points, trajectories, and even time. In quantum mechanics, people were forced to employ indirect methods, such as quantum entanglement, quantum phase transition, and quantum chaos, to explore bifurcation behavior in quantum world. On the contrary, the usage of nonlinear quantum dynamics developed in Section 2 allows us to analyze bifurcation from the conventional viewpoint. To illustrate how bifurcation emerges from quantum entanglement, we consider an entangled state for a “free” particle

$$\psi(x) = Ae^{\mathrm{i}\sqrt{2mE}x/\hbar} + Be^{-\mathrm{i}\sqrt{2mE}x/\hbar},\quad A,\ B\neq 0,\ x\in\mathbb{C}\,. \tag{26}$$

Depending on the bifurcation parameter, $\alpha = A/B \geq 0$, a free particle may move right or left, or merely oscillate between the two directions. To determine which case actually happens, we need the velocity information of the particle, obtained by substituting Eq.(26) into Eq.(10):

$$\frac{d\bar{x}}{d\bar{t}} = \frac{\alpha e^{i\bar{x}} - e^{-i\bar{x}}}{\alpha e^{i\bar{x}} + e^{-i\bar{x}}}, \quad \bar{x}(\bar{t}_0) = \bar{x}_0 \in \mathbb{C}, \tag{27}$$

where the dimensionless variables are defined as $\bar{x} = (\sqrt{2mE}/\hbar)x$ and $\bar{t} = (2E/\hbar)t$. The integration of Eq.(27) gives the following simple result:

$$\alpha e^{i\bar{x}} - e^{-i\bar{x}} = \beta e^{i\bar{t}}, \quad \alpha = \frac{1+\dot{\bar{x}}_0}{1-\dot{\bar{x}}_0} e^{-2i\bar{x}_0}, \quad \beta = \frac{2\dot{\bar{x}}_0}{1-\dot{\bar{x}}_0} e^{-i\bar{x}_0}. \tag{28}$$

Bifurcation occurs at the critical value of α

$$|\alpha| = |\beta|^2 / 4. \tag{29}$$

In the complex plane spanned by all the initial velocities $\dot{\bar{x}}_0 = \dot{\bar{x}}_R(0) + i\,\dot{\bar{x}}_I(0) \in \mathbb{C}$, the complex point $\dot{\bar{x}}_0$ satisfying the inequality (29) falls on the hyperbola:

$$\dot{\bar{x}}_R^2(0) - \dot{\bar{x}}_I^2(0) = 1/2. \tag{30}$$

According to the given initial velocity, we now can categorize the free-particle motion in the entangled state into three branches: (1) $\dot{\bar{x}}_R(0) > \sqrt{\dot{\bar{x}}_I^2(0) + 1/2}$ represents a particle moving right without bound; (2) $\left|\dot{\bar{x}}_R(0)\right| \le \sqrt{\dot{\bar{x}}_I^2(0) + 1/2}$ represents a particle oscillating within a bound interval, and (3) $\dot{\bar{x}}_R(0) < -\sqrt{\dot{\bar{x}}_I^2(0) + 1/2}$ represents a particle moving left without bound. As a numerical verification, Fig. 3 illustrates the free-particle trajectories with the same initial position $\bar{x}(0) = 0$ but different initial velocities $\dot{\bar{x}}_0 = 1 + 0.6i$, $1 + 0.8i$, and $-1 + 0.6i$, relating to the above three branches of trajectory.

The multiple-path motion of a free particle passing a single slit is shown in Fig. 4. The incident particles have only horizontal velocity components when passing the center of the slit, i.e., assume $\dot{\bar{x}}_R(0) = 1$ and $\dot{\bar{y}}_R(0) = 0$. To see the effect of the imaginary part of the initial velocity, we change the value of $\dot{\bar{y}}_I(0)$ from $-i$ to i with increment $0.02i$. The projected trajectories on the real $\bar{x}_R - \bar{y}_R$ plane for each value of $\dot{\bar{y}}_I(0)$ are recorded in Fig. 4. Notice that the whole trajectories appear on the two-dimensional complex plane $\bar{x} - \bar{y}$ but what we can observe are their projections in the real plane $\bar{x}_R - \bar{y}_R$. As is shown in Fig. 4, the entire particles incident upon the single slit A have the same initial velocity $\dot{\bar{x}}_R(0) = 1$ and $\dot{\bar{y}}_R(0) = 0$ as viewed from the real plane but after passing the slit, they follow different paths to the point B due to the influence of the unobservable imaginary part of the initial velocities $\dot{\bar{x}}_I(0)$ and $\dot{\bar{y}}_I(0)$.

At first glance, the fact, that particles with the same initial conditions and governed by the same equation of motion (27) follow different trajectories, cannot be allowed and explained by deterministic mechanics; but this is not true. The key point lies on the notion that what we mean the same initial condition actually stands for the same real part of the initial condition, while the imaginary part of the initial condition may be different. As viewed from the complex domain, particles passing A and B all have different imaginary parts of velocity; but as viewed from the projected real space, it appears as the multi-path phenomenon that there are infinite many paths connecting A and B all satisfying with the same initial and terminal conditions.

Conclusions

The lack of well-defined concept of time, trajectories and dynamic equations has prohibited the use of nonlinear analysis from quantum mechanics for a long time. The complex-valued mechanics provides a bridge between the probabilistic interpretation in the real space and the deterministic interpretation in the complex space. Through this bridge, researchers in nonlinear science and classical mechanics can employ methods familiar to them to analyze quantum systems in the complex space, and to predict and verify various quantum phenomena by projecting the results of analysis into the real space. Table 1 summarizes the established correspondence between quantum mechanics and complex mechanics.

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Table 1. The counterparts of quantum mechanics in complex mechanics

	Elements in quantum mechanics	**Counterparts in complex mechanics**
1	Quantum operators	Complex variables [1,4]
2	Quantization	Residue theory: Contour integration is independent of the shape of the contours [9,11].
3	Tunneling motion	Classical motions with complex force and complex acceleration [11].
4	Orbital angular motion	Angular motion in complex space [12].
5	Spin angular momentum	Zero dynamics of orbital angular momentum [13].
6	Probability mean value	Time-average mean value [14,9]
7	Wave motions	Interaction between real and imaginary motions [10]
8	Atomic shell structures	The structure of complex quantum potential [5]
9	Probability flow	Potential flow on the complex plane [15].
10	Probability density	The magnitude of complex quantum potential [2,16].
11	Quantum chaos	Chaos in complex-valued nonlinear systems [8].
12	Anti-particle and superluminal propagation	Complex-valued spacetime [17]

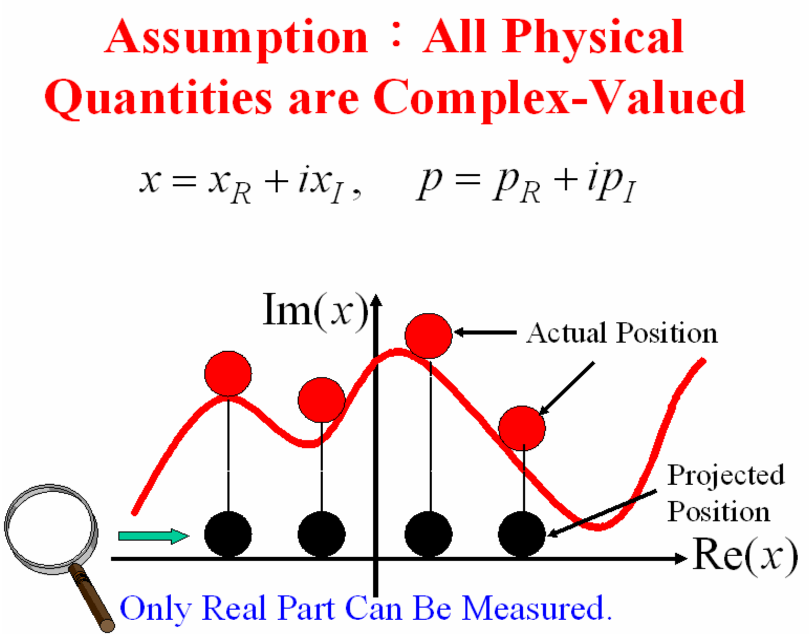

Fig. 1 Complex mechanics assumes that all physical quantities are complex-valued and only their real parts can be measured. The quantum phenomena manifesting in the real parts are caused by the influences from the imaginary parts.

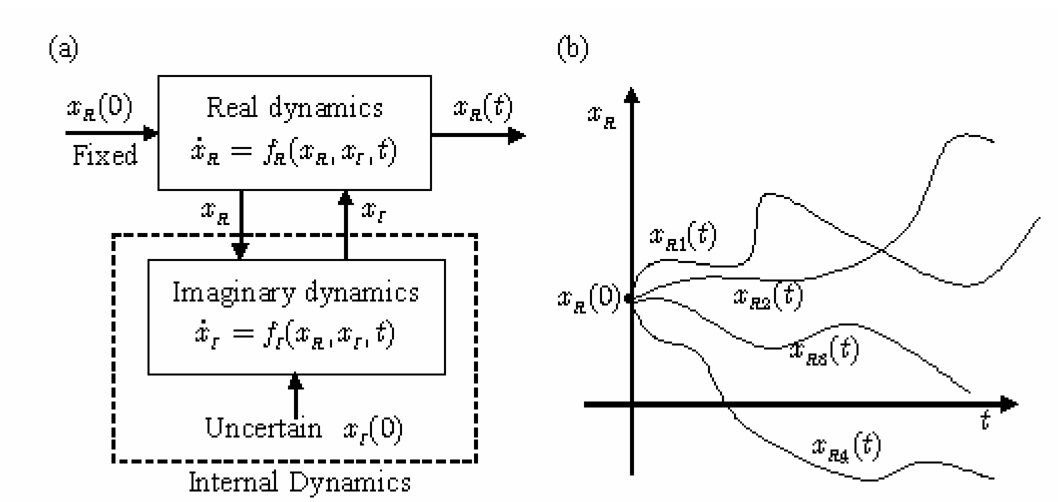

Fig. 2 The internal mechanism bringing forth multi-path behavior. (a) Infinitely many trajectories $x_R(t)$ are produced all with the fixed initial condition $x_R(0)$ via the interaction between the real dynamics and the imaginary mechanics by varying the value of $x_I(0)$. (b) A value of $x_I(0)$ generates a specific trajectory $x_R(t)$. Due to the unobservability of the imaginary dynamics $x_I(t)$, all of such trajectories, when viewed from the real space, start from the same initial position $x_R(0)$.

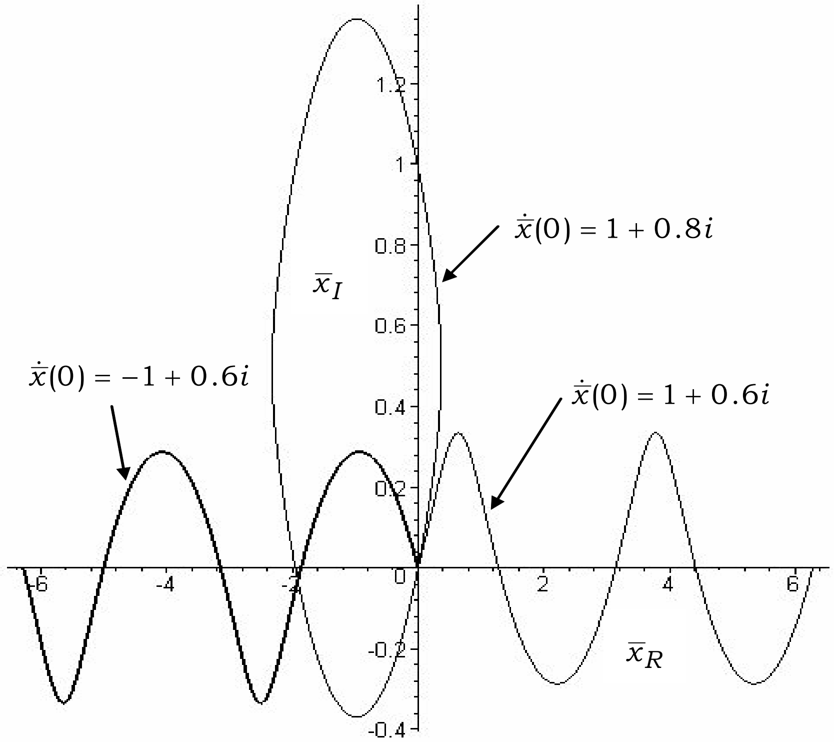

Fig. 3 Three types of trajectory for free particles with same initial position but different initial velocities.

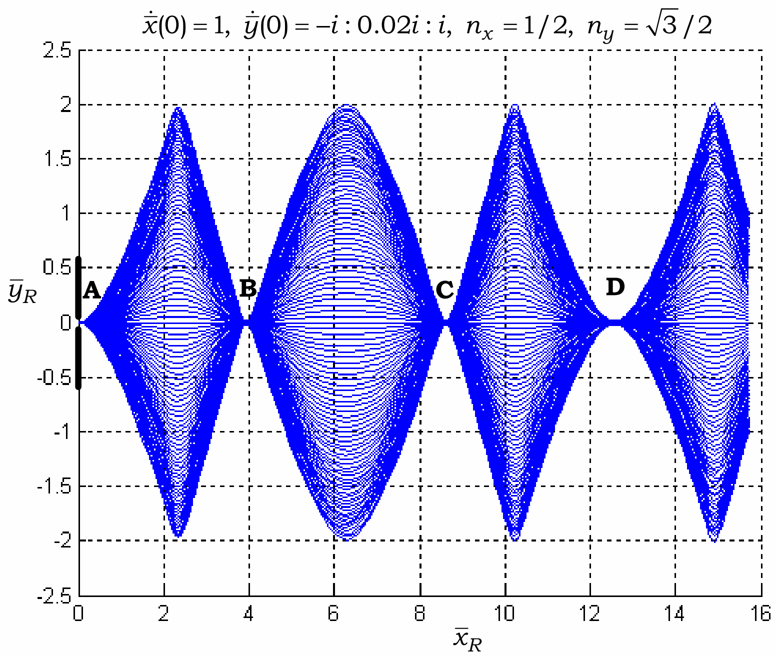

Fig. 4 The multi-path behavior for a free particle passing a single slit with horizontal incident velocity